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Weights and Robustness of Model-based Seasonal Adjustment

MARCO RIANI*
University of Parma, Italy

ABSTRACT
Model-based seasonal adjustment implicitly defines a set of weights at the ends of series as well as in the middle. Until now, with the exception of very simple models, the weights have been obtained numerically. In this paper we give the analytical expressions for the weights for both the structural and the ARIMA framework for a model which contains trend, seasonal and irregular component. In the final part of the paper we address the question of robustness of model-based seasonal adjustment. We analyse practically, using real time series, and theoretically, through the analysis of the shape of the weights, how the fitting of different specifications for the non-seasonal part affects the extraction of the seasonal component.

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KEY WORDS Kalman filter; Wiener–Kolmogorov filter; seasonal adjustment; signal extraction; Census X-11; basic structural model (BSM)

INTRODUCTION

The various procedures of seasonal adjustment can be divided into two major approaches: empirical and model based (MB). The former is based on repeated application of moving average filters (for example, the X-11 ARIMA method; see Dagum, 1980). The latter depends on a model for which the parameters are estimated by maximum likelihood (Bell and Hillmer, 1984). An overview of the advantages of the MB approach with respect to the use of empirical filters is given by Maravall (1996). Among MB procedures we can distinguish ARIMA (Hillmer and Tiao, 1982, henceforth AMB) and structural (Harvey, 1989—STS) methods. In AMB it is assumed that the series under study can be represented by an ARIMA model with constraints on the parameters. The (pseudo) spectrum of the estimated ARIMA model is generally split into three parts corresponding respectively to trend, seasonal, and irregular component. Some constraints are imposed in order to identify these components. Seasonal adjustment in STS is straightforward,

* Correspondence to: Marco Riani, Istituto di Statistica, Facoltà di Economia, University of Parma, via J. Kennedy 6, 43100 Parma, Italy.

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because the models are expressed directly as the sum of seasonal and other components. Once the model parameters have been estimated the seasonal part is removed by smoothing (Harvey *et al.*, 1997).

Seasonal adjustment can be regarded as a linear transformation of the data. MB seasonal adjustment implicitly defines a set of weights at the ends of the series as well as in the middle. The weights can be calculated by running the Kalman filter and smoother using a time series made up of zeros except at the point for which the weights are required, where the observation is set to one. Alternatively, the weights can be obtained numerically through the use of the Wiener–Kolmogorov filter (Box *et al.*, 1978; Maravall, 1994; Bianchi, 1996). In this paper, we obtain the analytical expressions for the weights for both the structural and the ARIMA framework for a model which contains trend, seasonal, and irregular component. These expressions are important because in this way (1) we can gain further insights into the differences and similarities between AMB and STS and (2) we can analyse the sensitivity of parameters to changes in the decay of the weight pattern. We also state the relationships between the parameters of the STS and those of AMB.

Using the time series of UK personal disposable income and non-durable consumption and the theoretical results found, we analyse the magnitude of the differences between AMB and STS. We also address the question of robustness of STS seasonal adjustment. More precisely, we analyse practically, using real time series, and theoretically, through the analysis of the shape of the weights, how the fitting of different specifications for the non-seasonal part affects the extraction of the seasonal component.

THEORETICAL COMPARISON BETWEEN STS AND AMB MODEL-BASED SEASONAL ADJUSTMENT METHODS

In all model-based seasonal adjustment procedures it is assumed that the observed time series (y_t) can be expressed as the sum of several orthogonal components representing respectively the trend, the seasonality, and the irregular white-noise term. For simplicity we consider a seasonal period $s = 2$. A structural time-series model with trend (μ_t), seasonal (γ_t), and irregular (ε_t), is defined as

$$y_t = \mu_t + \gamma_t + \varepsilon_t = \frac{\eta_t}{1 - L} + \frac{\omega_t}{1 + L} + \varepsilon_t \tag{1}$$

where η_t, ω_t and ε_t are three mutually and serially uncorrelated white-noise terms with zero means and variances equal to σ_η^2, σ_ω^2, σ_ε^2 respectively. While the STS approach starts by directly specifying the components and uses the Kalman filter for estimation, the AMB fits a model for the overall series and uses the Wiener–Kolmogorov filter. To identify the unobserved components, in AMB the stability of the trend and of the seasonal component is maximized creating artificial unit roots in the corresponding pseudospectra. In the standard terminology it is said that trend and seasonal are made canonical (Tiao and Hillmer, 1978). For example, the reduced form of model (1) can be written as follows:

$$(1 - L)(1 + L)y_t = (1 + \theta_1 L + \theta_2 L^2)a_t \tag{2}$$

where $a_t = WN(0,\sigma^2)$. The AMB approach decomposes this ARIMA specification into trend (p_t) seasonal (s_t) and irregular (d_t) in the following way:

$$y_t = p_t + s_t + d_t = \frac{1+L}{1-L} b_t + \frac{1-L}{1+L} c_t + d_t \tag{3}$$

where b_t, c_t, and d_t are three mutually and serially uncorrelated white-noise terms with zero mean and variance respectively equal to σ_b^2, σ_c^2, σ_d^2. The models for trend (p_t) and seasonal (s_t) contain in the numerator an artificial unit root in order to ensure that the variance of the irregular component is maximized.

By equating the autocovariances of the stationary forms of models (1) and (3) we obtain the relationships between the parameters which represent the variance of the trend and the seasonal:

$$\sigma_\eta^2 = 4\sigma_b^2 \quad \sigma_\omega^2 = 4\sigma_c^2 \tag{4}$$

For the irregular term, we have:

$$\sigma_\varepsilon^2 = \sigma_d^2 - \sigma_b^2 - \sigma_c^2 \quad \sigma_d^2 = \sigma_\varepsilon^2 + \frac{\sigma_\eta^2 + \sigma_\omega^2}{4} \tag{5}$$

Equation (5) shows that the variance of the structural irregular is always smaller than the canonical one. From equation (4) it is interesting to note that the variance of the structural trend (seasonal) is equal to four times the corresponding canonical one and these relationships do not involve other hyperparameters. This implies that the magnitude of the difference in the two seasonal adjusted (detrended) series is an increasing function of the variance of the corresponding unobserved component.

If in models (1) and (3) we consider separately the seasonal part from the non-seasonal one, we have the following equations:

$$y_t = \frac{(1+\theta L)}{1-L} \xi_t + \frac{\omega_t}{1+L} \quad \text{STS} \tag{6}$$

$$y_t = \frac{(1+\beta L)}{1-L} m_t + \frac{1-L}{1+L} c_t \quad \text{AMB} \tag{7}$$

In both cases the non-seasonal part is an ARIMA (0,1,1) process. Note that this is exactly true for any seasonal period. It is known (for example, Harvey, 1989, p. 68) that the parameter θ in equation (6) relates to the signal-to-noise ratio $q = \sigma_\eta^2/\sigma_\varepsilon^2$ as:

$$\theta = (\sqrt{q^2 + 4q} - q - 2)/2 \tag{8}$$

It is interesting therefore to analyse how the non-seasonal parameter β which appears in the AMB approach is linked to θ in order to have a better understanding of the differences and similarities of the two approaches. Equating the autocovariances of the stationary form of model (1) with those of model (7) it can be shown, after tedious but simple algebra, that:

$$\beta = (\sqrt{\bar{q}^2 + 4\bar{q}} - \bar{q} - 2)/2 \tag{9}$$

where $\bar{q} = 4\sigma_\eta^2/(4\sigma_\varepsilon^2 + \sigma_\omega^2)$. Comparing equation (8) with equation (9) we can immediately see that the parameter β can be interpreted in the same way as θ provided that we give a different definition to the signal-to-noise ratio q. Equation (9) shows that if the seasonal movements are slowly changing (i.e. $\sigma_\omega^2 \to 0$) then the two approaches tend to produce the same result. For a time series with a high variability in the seasonal component the AMB approach artificially transfers the seasonal movements in the non-seasonal part of the model. Usually, however, the variability in the seasonal component is low with respect to that of the trend and of the irregular, consequently these two approaches are not likely to give very different results.

Given these differences, it is interesting to analyse the shape of the weights of these two approaches (assuming a doubly infinite sample) in order to understand what determines their rate of decay and what are the main differences.

In the Appendix we prove that STS in order to obtain an estimate of the trend at time t ($\mu_{t|\infty}$) uses a weighted arithmetic mean of the observations in which the weight (W) applied to $y_{t\pm k}$ (lag k) satisfies the following equation:

$$W_k(\mu_{t|\infty}) = \frac{\lambda_1^k(\lambda_1+1)(\lambda_2-1) + \lambda_2^k(1-\lambda_1)(\lambda_2+1)}{(1-\lambda_1\lambda_2)(1-\lambda_1)(1-\lambda_2)(\lambda_2-\lambda_1)}\sigma_\eta^2/\sigma^2 \quad k = 0, \pm 1, \pm 2 \ldots \tag{10}$$

where λ_1 and λ_2 are linked to the parameters of the reduced form (model (2)) by the following relationships:

$$\lambda_1 + \lambda_2 = -\theta_1 \tag{11}$$

$$\lambda_1\lambda_2 = \theta_2 \tag{12}$$

Equating the autocovariances of model (1) with those of the reduced form (model (2)), it can be shown that $\theta_2 < 0$ so that λ_1 and λ_2 must have opposite sign. Without loss of generality we can assume $\lambda_1 < 0$ and $\lambda_2 > 0$. For the seasonal component (γ_t) we have:

$$W_k(\gamma_{t|\infty}) = \frac{\lambda_1^k(1-\lambda_1)(\lambda_2+1) + \lambda_2^k(1+\lambda_1)(\lambda_2-1)}{(1-\lambda_1\lambda_2)(1+\lambda_1)(1+\lambda_2)(\lambda_2-\lambda_1)}\sigma_\omega^2/\sigma^2 \qquad k = 0, \pm 1, \pm 2 \ldots \tag{13}$$

In the AMB approach for canonical trend we obtain:

$$W_k(p_{t|\infty}) = \frac{\lambda_1^{k-1}(\lambda_1+1)^3(\lambda_2-1) + \lambda_2^{k-1}(1+\lambda_2)^3(1-\lambda_1)}{(1-\lambda_1\lambda_2)(1-\lambda_1)(1-\lambda_2)(\lambda_2-\lambda_1)}\sigma_b^2/\sigma^2 \qquad k = \pm 1, \pm 2 \ldots \tag{14}$$

The central weight ($k = 0$) for canonical trend is given by:

$$W_0(p_{t|\infty}) = \frac{2(3+\lambda_1+\lambda_2-\lambda_1\lambda_2)}{(1-\lambda_1\lambda_2)(1-\lambda_1)(1-\lambda_2)}\sigma_b^2/\sigma^2 \tag{15}$$

The formula for the canonical seasonal is the following:

$$W_k(s_{t|\infty}) = \frac{\lambda_1^{k-1}(\lambda_1-1)^3(\lambda_2+1) - \lambda_2^{k-1}(1+\lambda_1)(\lambda_2-1)^3}{(1-\lambda_1\lambda_2)(1+\lambda_1)(1+\lambda_2)(\lambda_2-\lambda_1)}\sigma_c^2/\sigma^2 \qquad k = \pm 1, \pm 2 \ldots \tag{16}$$

The central weight ($k = 0$) for the canonical seasonal is given by:

$$W_0(s_{t|\infty}) = \frac{2(3 - \lambda_1 - \lambda_2 - \lambda_1\lambda_2)}{(1 - \lambda_1\lambda_2)(1 + \lambda_1)(1 + \lambda_2)}\sigma_c^2/\sigma^2$$

For the irregular in both approaches we have:

$$\frac{\lambda_1^{k-1}(1 - \lambda_1^2) - \lambda_2^{k-1}(1 - \lambda_2^2)}{(1 - \lambda_1\lambda_2)(\lambda_2 - \lambda_1)}\frac{\sigma_*^2}{\sigma^2} \quad \text{for } k = \pm 1, \pm 2 \ldots$$

and

$$\frac{2}{(1 - \lambda_1\lambda_2)}\frac{\sigma_*^2}{\sigma^2} \quad \text{for } k = 0 \tag{17}$$

where σ_*^2 is equal to σ_ε^2 or σ_d^2 according to whether we are using the STS or the AMB method.

Obtaining of the above formulae enables us to state the following propositions which we prove in the Appendix:

Proposition 1: if $\sigma_\eta^2 \to 0$, ($\sigma_b^2 \to 0$), then $\lambda_1 \to \theta_2$ and $\lambda_2 \to 1$.
Proposition 2: if $\sigma_\omega^2 \to 0$ ($\sigma_c^2 \to 0$), then $\lambda_1 \to -1$ and $\lambda_2 \to -\theta_2$.
Proposition 3: central weight for trend and seasonal is greater in STS than in AMB.
Proposition 4: at lag 1 the weight for trend is greater in STS than in AMB if $\lambda_2 > |\lambda_1|$.
Proposition 5: at lag 1 in both approaches the weight for trend is always positive.
Proposition 6: at lag 1 the weight for irregular is negative if $\sigma_\eta^2 > \sigma_\omega^2(\sigma_b^2 > \sigma_c^2)$.
Proposition 7: if $\sigma_\varepsilon^2 \to 0$, then

$$\lambda_1 \to -\theta_1 \to -\frac{\sqrt{\sigma_\eta^2} - \sqrt{\sigma_\omega^2}}{\sqrt{\sigma_\eta^2} + \sqrt{\sigma_\omega^2}}$$

and $\lambda_2 \to \theta_2 \to 0$. Structural weights for trend and seasonal therefore become:

$$W_k(\mu_{t|\infty}) = \frac{\lambda_1^{k-1}(1 + \lambda_1)}{(1 - \lambda_1)}\frac{\sigma_\eta^2}{\sigma^2} \quad k = \pm 1, \pm 2 \ldots \tag{18}$$

$$W_k(\gamma_{t|\infty}) = -\frac{\lambda_1^{k-1}(1 - \lambda_1)}{(1 + \lambda_1)}\frac{\sigma_\omega^2}{\sigma^2} \quad k = \pm 1, \pm 2 \ldots \tag{19}$$

In the last case the variance of the reduced form σ^2 is equal to $\left(\sqrt{\sigma_\eta^2} + \sqrt{\sigma_\omega^2}\right)^2$. Finally, the central weights ($k = 0$) become:

$$W_0(\mu_{t|\infty}) = \frac{2}{1 - \lambda_1}\frac{\sigma_\eta^2}{\sigma^2} \quad W_0(\gamma_{t|\infty}) = \frac{2}{1 + \lambda_1}\frac{\sigma_\omega^2}{\sigma^2}$$

Proposition 1 affirms that λ_2 and λ_1 are linked to the stability of the trend and the seasonal component respectively. If the movements in the trend (seasonal) tend to become deterministic $\lambda_2 \to 1(\lambda_1 \to -1)$.

In AMB when the trend and seasonal are made canonical some variability of these components is transferred to the irregular, therefore (as Proposition 3 states) the central weight given to the trend and seasonal is always less in AMB than in STS. Proposition 4 affirms that at lag 1 the weights given to the trend component in STS are greater than in AMB if $\sigma_\eta^2 < \sigma_\omega^2 (\sigma_b^2 < \sigma_c^2)$. Usually the movements in the trend are larger than those in the seasonal, therefore in practice at lag 1, the canonical trend uses a greater weight than the structural trend. This reflects the constraint of smoothness imposed on trend by the AMB approach.

Proposition 5 is of crucial importance because it states that even if the variance of the trend movements is much greater than that of the other components the extracted trend cannot fluctuate up and down.

Proposition 6 states that when the movements in the trend are greater than those in the seasonal, the irregular has the typical white-noise behaviour. In contrast to what happens for the trend, however, the weights at lag 1 for the irregular do not have a specific constraint. This implies that the extracted irregular component can deviate significantly from the pure white noise. It is known that in unobserved component models the use of MMSE estimators implies that the autocovariance-generating function of the irregular component is equal to the inverse of that of the original model. This also explains why the formulae for the weights for the irregular are the same in both STS and AMB. The flexibility in the pattern of the weights for the irregular component is very important because it implies that if some unobserved components are mistakenly not inserted in the model the effect of these neglected components is likely to be included in the irregular term. In the fourth section, when we examine the robustness of MB seasonal adjustment, we will be able to give more support to this intuition.

It is easy to see that in both approaches the rate of decay of the weights, when all three parameters are greater than zero, is determined by a difference equation governed by the two parameters λ_1 and λ_2. If there are just two hyperparameters greater than zero then (as equations (18) and (19) show) the rate of decay is a function of their ratio.

After examining the properties of the weight functions for the unobserved components, it is interesting to study them graphically. We consider two situations of low and high variability in the trend and seasonal movements. In Figure 1 we can visually inspect the weights used to extract trend, seasonal, and irregular when $\sigma_\eta^2/\sigma_\varepsilon^2 = 1$ and $\sigma_\omega^2/\sigma_\varepsilon^2 = 0.01$. Figure 1 shows (confirming what has been proved in Propositions 3 and 4) that the shape of the weights of structural trend is larger but narrower with respect to canonical trend. In the seasonal component there does not seem to be an appreciable difference between the two approaches. It is interesting to know what happens when we increase the variability of the trend and of the seasonal movements. In Figure 2 we analyse the shape of the weights when $\sigma_\eta^2/\sigma_\varepsilon^2 = 10$ and $\sigma_\omega^2/\sigma_\varepsilon^2 = 1$. This figure shows that in this last case the two trends will show marked differences. More precisely the canonical trend will be much smoother than the STS one. In the seasonal (apart from the height of the central peak, which, as stated in Proposition 4, is higher in STS) even in the presence of large seasonal movements there does not seem to be an appreciable difference between AMB and STS. Finally, it is evident that in this case the central weight for the irregular is much greater in the canonical approach. This reflects the constraint of the maximization of the variance of the irregular component imposed by AMB.

In conclusion, these plots show that the main differences between STS and AMB refer to the trend and if the purpose of the analysis is seasonal adjustment, the two approaches are likely to produce very similar results.

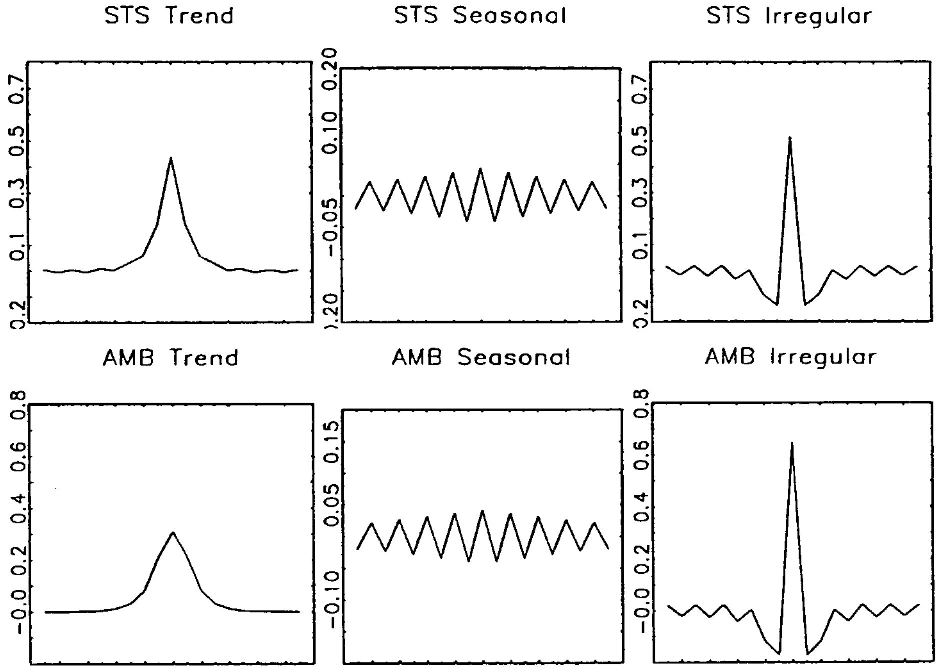

Figure 1. Comparison of weights for STS (*top*) and AMB (*bottom*) (slowly changing trend and seasonal)

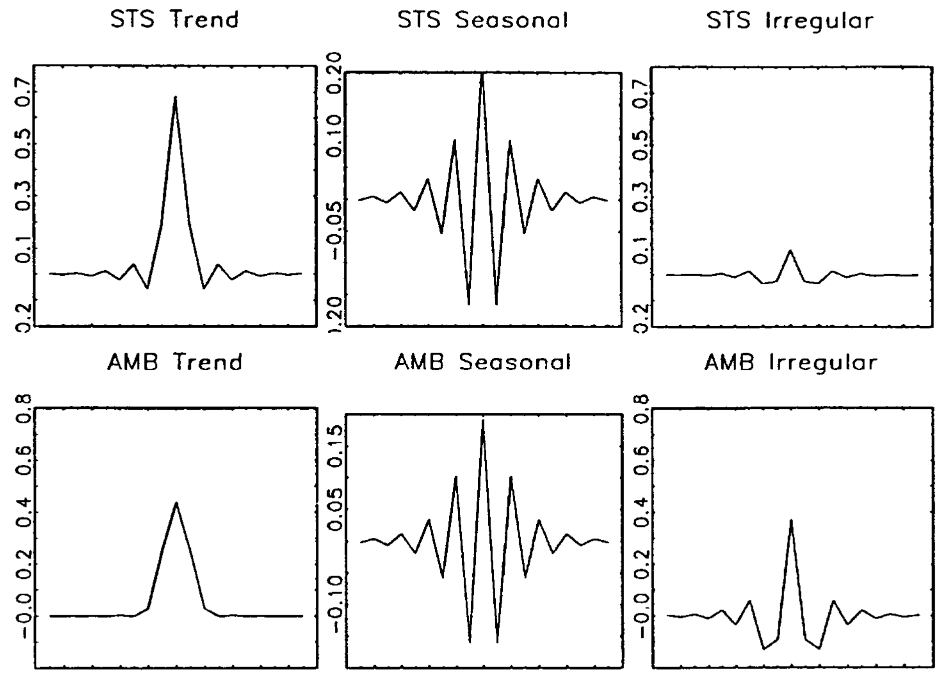

Figure 2. Comparison of weights for STS (*top*) and AMB (*bottom*) (unstable trend and seasonal)

PRACTICAL COMPARISON BETWEEN AMB AND STS

In the former section we have proved theoretically that unless the variability of the seasonal part is high with respect to the non-seasonal one, AMB and STS produce very similar seasonally adjusted series. In this section we analyse practically through the use of real time series how large are the differences between these two approaches. AMB is implemented in the program of Maravall and Gomez. This program is made up of two routines TRAMO (Gomez and Maravall, 1994) and SEATS (Maravall and Gomez, 1994). TRAMO initially "linearizes" the series, removing outliers and structural breaks and creates the input series for SEATS which performs the AMB seasonal adjustment. ARIMA models are not robust to the presence of atypical observations therefore a preliminary linearization is necessary. The class of structural models, in contrast, is limited with respect to the ARIMA ones, but it is more robust to the presence of outliers because of the constraints that structural models impose at the outset on the range of admissibility of the parameters.

The series we take into consideration are those of quarterly UK personal disposable income (PDI) and non-durable consumption (CONS). The period we consider goes from the first quarter of 1955 to the second quarter of 1993. For UK PDI, TRAMO considers observations referred to the first quarter of 1966 and to the last quarter of 1979 as outliers and removes them. In addition, TRAMO detects a structural break from the second quarter of 1972 to the third quarter of 1976. For concerns non-durable consumption, TRAMO linearizes this series in the first quarter of 1968 and from the first quarter of 1973 to the fourth quarter of 1983.

On the two linearized series SEATS estimates the following ARIMA models:

$$\Delta\Delta_4 \ln y_t = (1 - 0.17L)(1 - 0.64L^4)a_t \quad \text{PDI}$$

$$\Delta\Delta_4 \ln y_t = (1 - 0.39L^4)a_t \quad \text{CONS}$$

STS seasonal adjustment is implemented in BSTAMP. This program is a batch version of STAMP (Koopman *et al.*, 1995) especially directed towards seasonal adjustment. It can cope with trading days, transformations, and Easter effects. It automatically removes outliers only if the corresponding residuals are greater than a certain threshold. This routine uses as default the basic structural model[1] with trigonometric seasonality. In the former series we could have easily removed the outliers but we preferred to estimate the model without preliminary modifications of the data in order to analyse the sensitivity of the results to the presence of these atypical observations. Our suggestion is that a preliminary correction of the series is unnecessary unless there are observations which are quite extreme.

BSTAMP gave the following estimates:

$$\begin{aligned} \frac{\sigma_\omega^2}{\sigma_\eta^2} &= 0.0093 \quad \frac{\sigma_\zeta^2}{\sigma_\eta^2} = 0 \quad \frac{\sigma_\varepsilon^2}{\sigma_\eta^2} = 0.1872 \quad \text{PDI} \\ \frac{\sigma_\omega^2}{\sigma_\eta^2} &= 0.0180 \quad \frac{\sigma_\zeta^2}{\sigma_\eta^2} = 0.0339 \quad \frac{\sigma_\varepsilon^2}{\sigma_\eta^2} = 0.0686 \quad \text{CONS} \end{aligned} \tag{20}$$

[1] The basic structural model (BSM) is defined by a local linear trend in which the variance of the level and the slope are denoted by σ_η^2 and σ_ζ^2, a seasonal component and an irregular term (Harvey, 1989, p. 47).

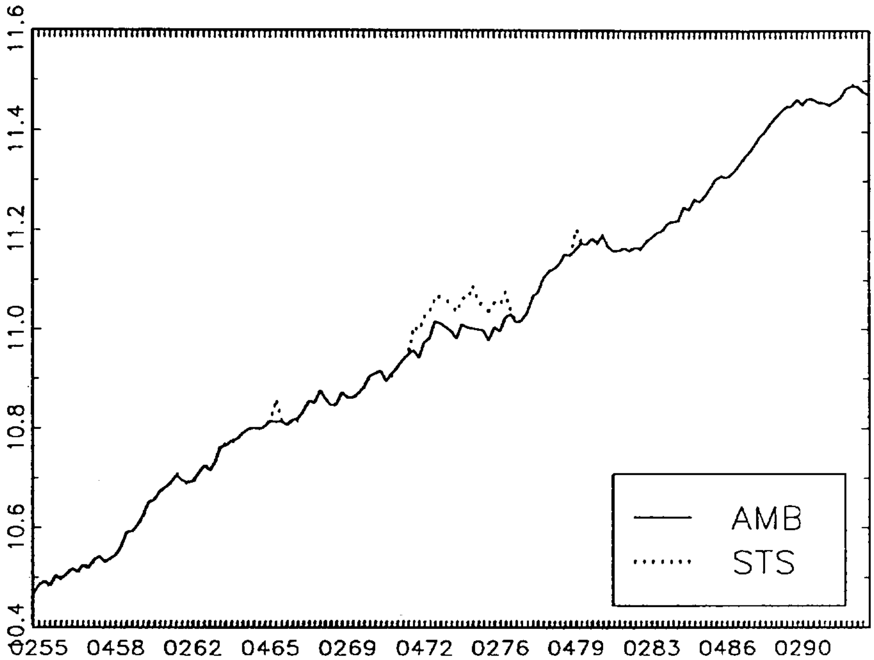

Figure 3. UK seasonal adjusted series of personal disposable income using TRAMO-SEATS and BSTAMP

The variability in the level of the trend seems to dominate that of the other components. In addition, the variability in seasonal movements seems to be more marked in the UK series of consumption than in that of PDI. From an examination of Figures 3 and 4 which show the seasonally adjusted series obtained by these two approaches, it appears that the output of BSTAMP is hardly distinguishable from that of SEATS. There is a marked difference only in the periods in which there has been the linearization of TRAMO. This suggests that if the focus of the analysis is on seasonal adjustment, the differences between these two approaches are likely to be small for many real data sets.

ANALYSIS OF ROBUSTNESS OF MODEL BASED SEASONAL ADJUSTMENT

In this section we address the problem of robustness of MB seasonal adjustment. For this purpose we use the UK quarterly time series of PDI of the former section. After the introduction of a structural cycle (see Harvey, 1989, p. 39) with variance σ_k^2 the estimated hyperparameters become:

$$\frac{\sigma_\omega^2}{\sigma_k^2} = 0.0195 \quad \frac{\sigma_\eta^2}{\sigma_k^2} = 0.5915 \quad \frac{\sigma_\zeta^2}{\sigma_k^2} = 0 \quad \frac{\sigma_\varepsilon^2}{\sigma_k^2} = 0.8492 \tag{21}$$

If we compare these hyperparameters with those of the model without a cycle (model (20)) we can see that the introduction of a cycle seems to affect considerably the ratios among some hyperparameters. If we inspect graphically the series with the estimated trend in a basic structural

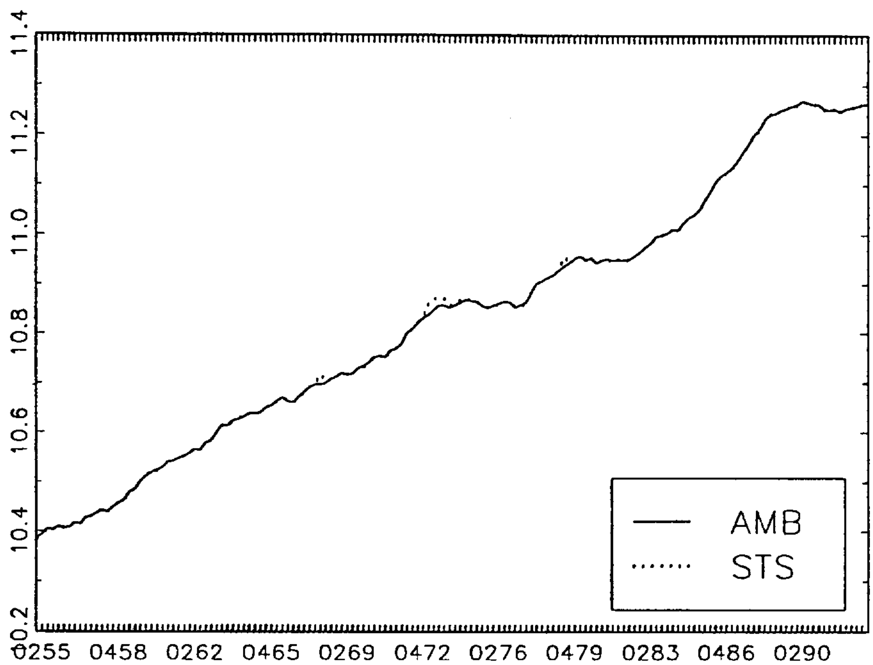

Figure 4. UK seasonal adjusted series of non-durable consumption using TRAMO-SEATS and BSTAMP

model (Figure 5) and the same series with the trend once a structural cycle has been included in the model (Figure 6), we can see a significant difference in the estimated trend. In other words, the introduction of a stochastic cycle results in a smooth trend. Running the Kalman filter on a time series made up of zeros and ones and using the ratio of the hyperparameters found earlier we can easily compute the weights used to estimate the underlying components of the time series.

Figure 7 reports the weights used to extract the trend and the irregular component. The left part refers to a model without a cycle, the right part to a specification with a cycle. While the two graphs on top refer to the trend, those at the bottom concern the irregular. When no cycle is present the weights for trend are characterized by a sharp central peak (time t) and a monotonically decreasing pattern. When a structural cycle is added, the height of the peak consistently reduces and we have two local maxima corresponding to times $t \pm p/2$ where p is the period of the cycle (this guarantees a smooth trend). Conversely, the weights for cycle, not reported here, show two local minima at times $t \pm p/2$. As regards the irregular, the shape of the weights does not appreciably change after the introduction of a cycle but the height of the central peak increases in a model with cycle. In this last case the trend is smooth so if an irregular is present, it will absorb some of the variability that was formerly in the trend. It is worth noting (as shown theoretically in the previous section) that, irrespective of the presence of a cyclical component, the weights for trend are always positive at lag 1.

As emerges from the examination of Figure 8, which reports the seasonal component estimated in a model without a cycle (top) and with a cycle (bottom), we can clearly see that the introduction of the cyclical component does not seem to affect the seasonal component (the maximum

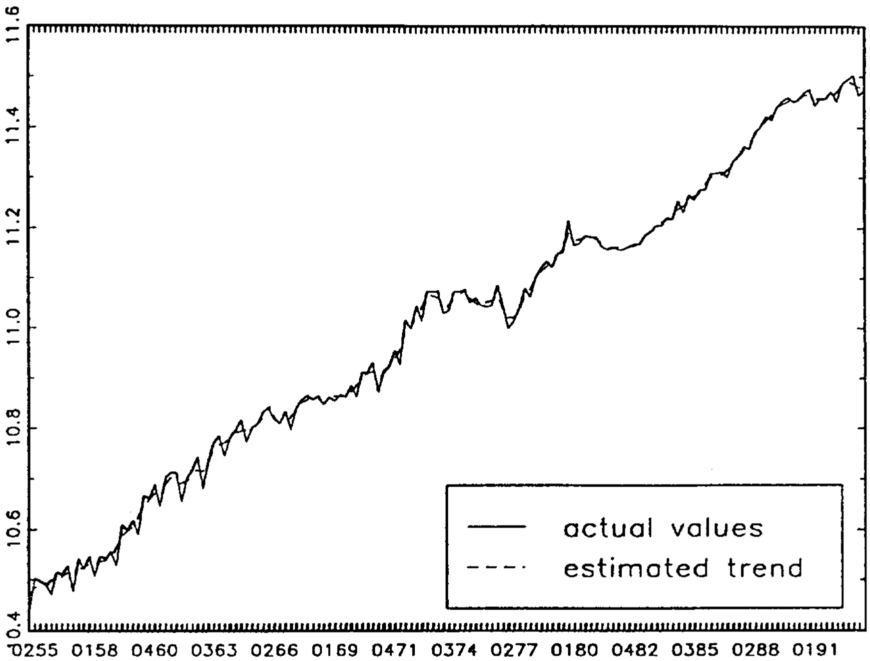

Figure 5. Estimated trend in UK PDI using a BSM

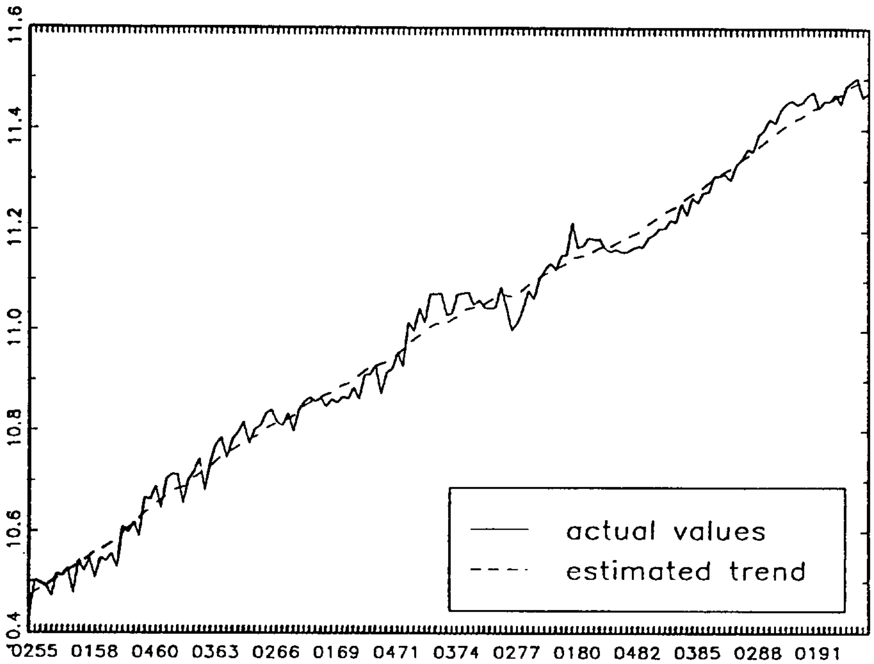

Figure 6. Estimated trend in UK PDI using a BSM + cycle

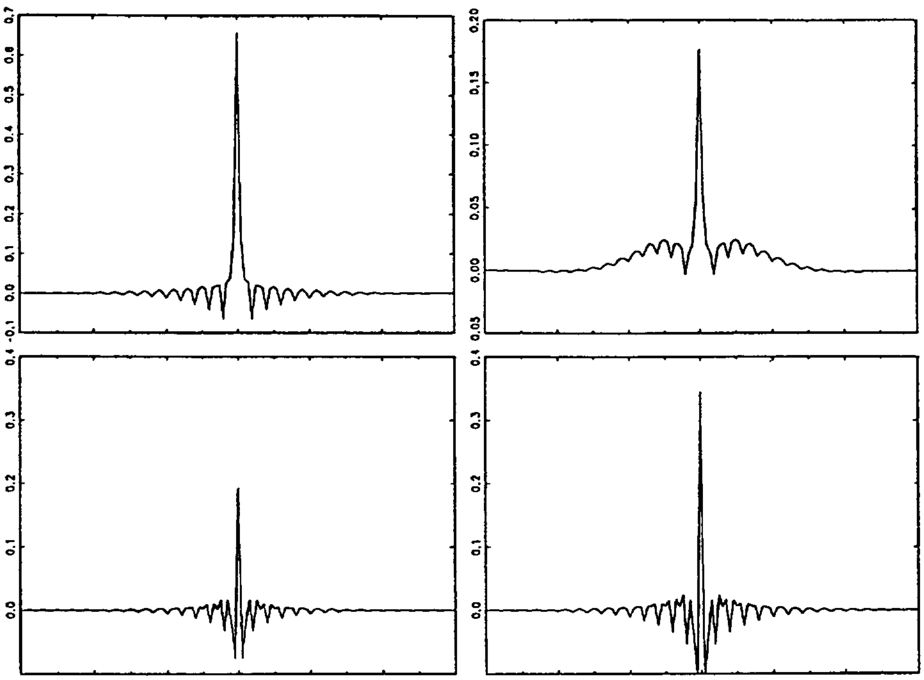

Figure 7. Weights used to estimate the trend and the irregular in UK PDI. In the two graphs at the top we can find the weights for trend, in those at the bottom the weights for the irregular component. The graphs on the left refer to a BSM, those on the right to a BSM + cycle

absolute difference between the two seasonal components is 0.002). This result is very important from a practical point of view because it says that if the purpose is seasonal adjustment the estimate of the seasonal component is practically unaffected by the introduction (omission) of a cycle.

Remark: Another question of interest is the following. How is the estimated trend affected by a model which does not consider the seasonal component? From the analysis we conducted (not reported here for lack of space) on several seasonal time series it emerged that in a model trend plus irregular the seasonal component is entirely included in the second component. The flexibility in the weights of the irregular term compared to the rigidity of the weights of trend generally guarantees that the omitted part of the model is included in ε_t.

CONCLUSIONS

In this paper we have analysed the differences and similarities of the two leading approaches of seasonal adjustment: the one which uses ARIMA models and the second which starts from structural models. We have shown theoretically that unless the variation in the seasonal component is very large with respect to the other parts of the model, these two approaches give very similar results. This part is illustrated in the third section, using real time series which display a different degree of variation in the seasonal movements.

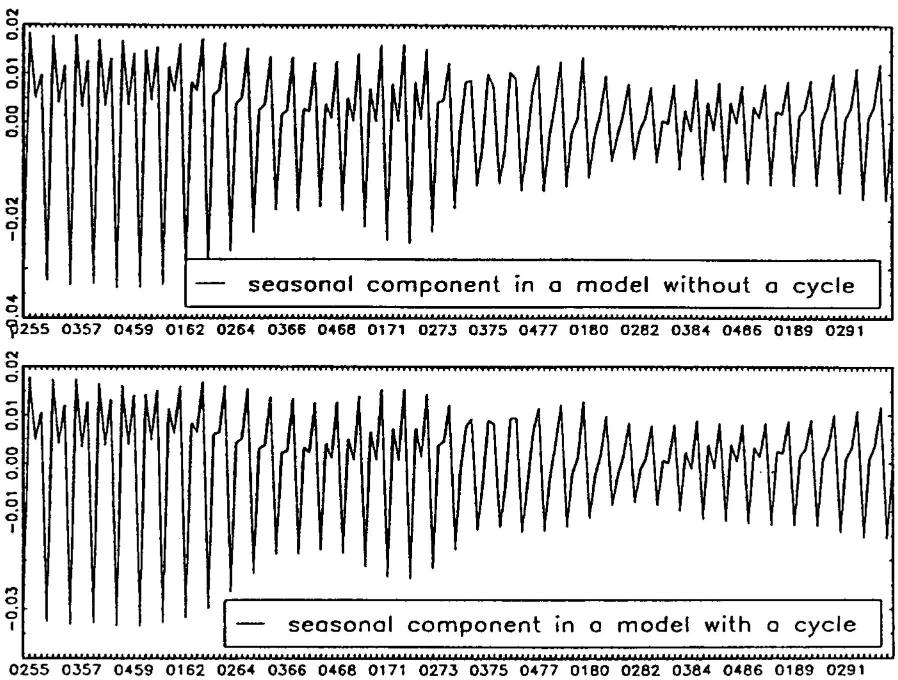

Figure 8. Seasonal component of UK PDI from a BSM (*top*) and a BSM + cycle (*bottom*)

In the fourth section we have seen that seasonal adjustment in a structural model is relatively robust with regard to model specification. For example, the omission (inclusion) of a stochastic cycle has little effect on the estimate of the seasonal component.

APPENDIX

Proof of equations (10)–(16)

We will prove only equation (10): the other expressions follow similar arguments. Using the Wiener–Kolmogorov filter the minimum mean square error estimator of trend ($\mu_{t|\infty}$) in equation (1) is given by the following equation (Bell, 1984):

$$
\begin{aligned}
\mu_{t|\infty} &= \frac{\dfrac{\sigma_\eta^2}{(1-L)(1-F)}}{\dfrac{\sigma_\eta^2}{(1-L)(1-F)} + \dfrac{\sigma_\omega^2}{(1+L)(1+F)} + \sigma_\varepsilon^2} y_t \\
\mu_{t|\infty} &= \frac{(1+L)(1+F)}{\sigma^2(1+\theta_1 L+\theta_2 L^2)(1+\theta_1 F+\theta_2 F^2)} \sigma_\eta^2 y_t \qquad \text{(A1)} \\
&= \frac{(1+L)(1+F)}{\sigma^2(1-\lambda_1 L)(1-\lambda_2 F)(1-\lambda_1 L)(1-\lambda_1 F)} \sigma_\eta^2 y_t
\end{aligned}
$$

where λ_1 and λ_2 are defined as in equations (11) and (12) and σ^2 is the variance of the reduced-form model. Bringing to the numerator the denominator of equation (22) we have:

$$\mu_{t|\infty} = \sum_{r=0}^{\infty}(\lambda_1 L)^r \sum_{r=0}^{\infty}(\lambda_1 F)^r \sum_{r=0}^{\infty}(\lambda_2 L)^r \sum_{r=0}^{\infty}(\lambda_2 F)^r (1+L)(1+F)(\sigma_\eta^2/\sigma^2)y_t$$

$$\mu_{t|\infty} = \Big\{(1+\lambda_1\lambda_2) + (\lambda_1+\lambda_2)(L+F) + [(\lambda_1^2+\lambda_2^2) + (1-\lambda_1\lambda_2)\lambda_1\lambda_2](L^2+F^2) + \cdots$$
$$+ [(\lambda_1^k+\lambda_2^k) + (1-\lambda_1\lambda_2)\sum_{l=1}^{k-1}\lambda_1^l\lambda_2^{k-l}](L^k+F^k) + \cdots\Big\}$$
$$\times (1+L)(1+F)(\sigma_\eta^2/\sigma^2)y_t/[(1-\lambda_1\lambda_2)(1-\lambda_1^2)(1-\lambda_2^2)]$$

After some manipulations it is found that the coefficient of L^k (F^k) ($k = 0,1, \ldots$) satisfies the following equation:

$$W_k(\mu_{t|\infty}) = \frac{(1+\lambda_1)(1+\lambda_2)(\lambda_1^k+\lambda_2^k - \sum_{l=1}^{k}\lambda_1^l\lambda_2^{k-l+1} + \sum_{l=0}^{k-1}\lambda_1^l\lambda_2^{k-1-l})}{(1-\lambda_1\lambda_2)(1-\lambda_1^2)(1-\lambda_2^2)}(\sigma_\eta^2/\sigma^2)$$

From this expression after some simplifications and cancellations it is easy to obtain model (10).

Proof of Propositions 1, 2 and 7

Equating the autocovariances of model (1) with those of the reduced form (model (2)) we have the following system of equations:

$$\alpha(1+\theta_1^2+\theta_2^2) = a \quad \text{(A2)}$$

$$\alpha\theta_1(1+\theta_2) = b \quad \text{(A3)}$$

$$\alpha\theta_2 = -1 \quad \text{(A4)}$$

where $a = 2q_\eta + 2q_\omega + 2$, $b = q_\eta - q_\omega$, $\alpha = \sigma^2/\sigma_\varepsilon^2$, $q_\eta = \sigma_\eta^2/\sigma_\varepsilon^2$ and $q_w = \sigma_\omega^2/\sigma_\varepsilon^2$. Solving this system of equations θ_2 is found as the solution of the following fourth-degree equation:

$$\theta_2^4 + (2+a)\theta_2^3 + (2+b^2+2a)\theta_2^2 + (2+a)\theta_2 + 1 = 0$$

which gives:

$$\theta_2 = \frac{\sqrt{2}\sqrt{-(a+2)\sqrt{a^2-4a-4b^2+4} + a^2 - 2b^2 - 4} + \sqrt{a^2-4a-4b^2+4} - a - 2}{4} \quad \text{(A5)}$$

Parameter θ_1 is found by solving:

$$\theta_1 = \frac{b}{(1+\theta_2)\alpha} \quad \text{(A6)}$$

From equations (26) and (27) it is easy to see that if $q_\eta \to 0$ then

$$\theta_2 \to \frac{\sqrt{q_\omega(q_\omega + 4)} - q_\omega - 2}{2}$$

$$\theta_1 \to \frac{q_\omega - \sqrt{q_\omega(q_\omega + 4)}}{2}$$

Solving equations (11) and (12) after some algebra the result follows. The proofs of Proposition 2 and 7 follow similar arguments and are omitted.

Proof of Propositions 3–6

For example, as concerns Proposition 3 for the part which refers to the trend, from equation (10) when $k = 0$ and equation (15) we have:

$$\frac{(\lambda_1 + 1)(\lambda_2 - 1) + (1 - \lambda_1)(\lambda_2 + 1)}{\lambda_2 - \lambda_1}\sigma_\eta^2 > 2(3 + \lambda_1 + \lambda_2 - \lambda_1\lambda_2)\sigma_b^2$$

Using equation (4) after some manipulations we obtain:

$$4\sigma_\eta^2 > (3 + \lambda_1 + \lambda_2 - \lambda_1\lambda_2)\sigma_\eta^2 \qquad \text{(A7)}$$

Remembering that $(-1 < \lambda_1 < 0)$ and $(0 < \lambda_2 < 1)$ the result follows easily. The other propositions are proved similarly.

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Author's biography:
Marco Riani is Lecturer in Statistics, Istituto di Statistica, University of Parma. He holds a doctorate in Statistics from the University of Florence. His research interests include time series and the analysis of multivariate data.

Author's address:
Marco Riani, Istituto di Statistica, Facoltà di Economia, University of Parma, via J. Kennedy 6, 43100 Parma, Italy.

